GENERALIZATION OF THE BIOT VARIATIONAL EQUATION IN THE HEAT-CONDUCTION THEORY

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The Biot variational method is extended to the case of finite rate of thermal momentum propagation. A generalized Lagrange equation is obtained which corresponds to an equation of nonstationary heat conduction of hyperbolic type.

At this time the Biot variational method [1] is used sufficiently extensively for the approximate solution of nonstationary heat conduction and convective heat-exchange problems.

In particular, the Biot variational equation is used to describe the nonstationary heat conduction in an incompressible material in the form [1]

$$\frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i, \tag{1}$$

where

$$V = V(q_1, q_2, \ldots, q_n) = \frac{\rho c}{2} \int_{V_0} T^2 dV_0$$

is the "temperature potential";

$$D = \frac{1}{2\lambda} \int_{V_0} \dot{H}^2 dV_0$$

is the "dissipation function"; $\dot{\mathbf{H}} = \partial \mathbf{H}/\partial t = \mathbf{J}_q$ is the thermal flux density vector; and

$$Q_i = -\int_{S_0} T \frac{\partial \mathbf{H}}{\partial q_i} \mathbf{n} dS_0$$

is the so-called "thermal force" which is used to take account of the boundary conditions of the problem. Assigning the energy conservation law

$$\rho c \frac{\partial T}{\partial t} = -\operatorname{div} \mathbf{J}_q \tag{2}$$

and using the Fourier hypothesis

$$-\lambda \operatorname{grad} T = \mathbf{J}_q \tag{3}$$

corresponds to the variational equation (1).

The last relationship is based on the assumption of an infinite rate of thermal momentum propagation. In the more general case, the interrelation between the thermal flux density and the temperature gradient is expressed by the formula [2, 3]

$$-\lambda \operatorname{grad} T = \mathbf{J}_q + t_r \frac{\partial \mathbf{J}_q}{\partial t}, \tag{4}$$

corresponding to a finite rate of thermal momentum propagation $v_T = (\lambda/\rho c t_r)^{1/2}$. The equation of nonstationary heat conduction in an incompressible material hence becomes

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$$\rho c\left(\frac{\partial T}{\partial t}+t_r\frac{\partial^2 T}{\partial t^2}\right)=\operatorname{div}(\lambda\operatorname{grad}T). \tag{5}$$

The generalized phenomenological Vernotte–Lykov relationship (4) has recently attracted the attention of researchers [4–7]. In this connection, the question arises as to how the Biot variational equation (1) is transformed if the more general relationship (4) is used in place of the Fourier hypothesis (3). To answer this question, we convert relationship (4) by introducing the field vector **H**:

$$\frac{1}{\lambda}\left(\frac{\partial \mathbf{H}}{\partial t}+t_r\frac{\partial^2 \mathbf{H}}{\partial t^2}\right)+\operatorname{grad}T=0. \tag{6}$$

Multiplying both sides of this equation by the variation $\delta\mathbf{H}$ and integrating over the body volume V_0, we obtain

$$\int_{V_0}\operatorname{grad}T\delta\mathbf{H}\,dV_0+\int_{V_0}\frac{1}{\lambda}\left(\frac{\partial \mathbf{H}}{\partial t}+t_r\frac{\partial^2 \mathbf{H}}{\partial t^2}\right)\delta\mathbf{H}dV_0=0. \tag{7}$$

Integrating the first term by parts and using the Ostrogradskii–Gauss formula, we find

$$-\int_{V_0}T\delta(\operatorname{div}\mathbf{H})\,dV_0+\int_{V_0}\frac{1}{\lambda}\left(\frac{\partial \mathbf{H}}{\partial t}+t_r\frac{\partial^2 \mathbf{H}}{\partial t^2}\right)\delta\mathbf{H}dV_0=-\int_{S_0}T\mathbf{n}\delta\mathbf{H}dS_0, \tag{8}$$

where $\mathbf{n}$ is the unit vector of the external normal to the surface S_0 bounding the volume V_0.

Upon insertion of the vector **H**, the energy conservation equation (2) becomes

$$\rho c\,(T-T_0)=-\operatorname{div}\mathbf{H}. \tag{9}$$

Taking (9) into account, we write (8) in the form

$$\rho c\int_{V^0}T\delta T dV_0+\frac{1}{\lambda}\int_{V_0}\left(\frac{\partial \mathbf{H}}{\partial t}+t_r\frac{\partial^2 \mathbf{H}}{\partial t^2}\right)\delta\mathbf{H}dV_0=-\int_{S_0}T\mathbf{n}\delta\mathbf{H}dS_0, \tag{10}$$

where it is assumed that ρc = const and λ = const.

Let us assume that the functions T and H depend on certain generalized coordinates q_i ($i = 1, 2, \ldots, n$) such that $T = T(q_i)$, $H = H(q_i)$u

$$\delta T=\sum_{i=1}^{n}\frac{\partial T}{\partial q_i}\delta q_i,\qquad \delta\mathbf{H}=\sum_{i=1}^{n}\frac{\partial \mathbf{H}}{\partial q_i}\delta q_i. \tag{11}$$

Let us introduce the functions $v = v(q_i)$ and $k = k(q_i, \dot{q}_i)$ by the formulas

$$v=\frac{\rho c}{2}T^2,\qquad k=\frac{1}{2\lambda}\left(\frac{\partial \mathbf{H}}{\partial t}\right)^2. \tag{12}$$

Let us find the partial derivatives of the functions v and k with respect to the generalized coordinates and the generalized velocities

$$\frac{\partial v}{\partial q_i}=\rho cT\frac{\partial T}{\partial q_i},\quad \frac{\partial k}{\partial q_i}=\frac{1}{\lambda}\dot{H}\frac{\partial \dot{H}}{\partial q_i},\quad \frac{\partial k}{\partial \dot{q}_i}=\frac{1}{\lambda}\dot{H}\frac{\partial \dot{H}}{\partial \dot{q}_i},\quad \frac{d}{dt}\frac{\partial k}{\partial \dot{q}_i}-\frac{\partial k}{\partial q_i}=\frac{1}{\lambda}\frac{\partial \mathbf{H}}{\partial q_i}\cdot\frac{\partial^2\mathbf{H}}{\partial t^2}.$$

Using these expressions, we rewrite the variational equation (10)

$$\sum_{i=1}^{n}\left\{\int_{V_0}\frac{\partial v}{\partial q_i}dV_0+t_r\int_{V_0}\left[\frac{d}{dt}\left(\frac{\partial k}{\partial \dot{q}_i}\right)-\frac{\partial k}{\partial q_i}\right]dV_0+\int_{V_0}\frac{\partial k}{\partial \dot{q}_i}dV_0\right\}\delta q_i=\sum_{i=1}^{n}Q_i\delta q_i, \tag{13}$$

where we used the notation $Q_i=-\int_{S_0}T\mathbf{n}\frac{\partial \mathbf{H}}{\partial q_i}dS_0$.

If the variations of the generalized coordinates (δq_i) are mutually independent, then (13) decomposes into n equations of variational type

$$\frac{\partial V}{\partial q_i}+t_r\left[\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right)-\frac{\partial K}{\partial q_i}\right]+\frac{\partial K}{\partial \dot{q}_i}=Q_i\ \ (i=1,\ 2,\ \ldots,\ n), \tag{14}$$

where it is assumed

$$V = \int_{V_0} v dV_0, \quad K = \int_{V_0} k dV_0. \tag{15}$$

Comparing (1) and (14), the deduction can be made that using the Vernotte–Lykov relationship (4) in place of the Fourier hypothesis (3) would result in the appearance of an additional member in the Biot variational equation. Let us note that this additional member agrees in form (see the expression in the square brackets in the generalized equation (14)) with the corresponding component of the Lagrange equation in analytical mechanics, which takes account of the influence of the kinetic energy of the mechanical system [8]. For $t_r = 0$ and $K = D$ Eq. (14) goes over into the Biot equation (1).

NOTATION

T, temperature; t, time; t_r, relaxation time; x, y, z, spatial coordinates; ρ, c, λ, coefficients of volume density, specific heat, and heat conductivity of the material; q_i, generalized coordinates; the dot above the variables q_i and H denotes differentiation with respect to time; V_0 = const, S_0 = const are the body volume and surface area bounding this volume, respectively.

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EFFECT OF SLITS ON THE RESISTANCE OF A CONDUCTING FILM

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Formulas are derived for the resistance offered to a steady or quasisteady current by a conducting film with straight and Π-shaped slits.

One way of changing the resistance of a conducting film used as a resistor is to cut slits in it. A slit is a thin curve along which a conducting layer has been removed, i.e., in slits the conductivity $\sigma = 0$.

The resistance of a film with a slit can be calculated by using the familiar analogy between the hydrodynamics problem of flow past a certain body and the electrostatics problem [1], and also the analogy between the electrostatics problem and the problem of the distribution of a steady or quasisteady current [2].

1. We calculate the change in resistance of a film with a straight slit perpendicular to the lines of flow at infinity (Fig. 1). We denote the width of the film by $2l$ and the length of the slit by 2b. We assume that $l \gg b$ and that the slit is located in the middle of the film so that the perturbing effect of its ends does not extend to the edges of the film. The solution of the corresponding hydrodynamics problem of the flow of a fluid past a plate in an infinite medium is given in [1]. For the problem under consideration, we write the conformal mapping function in the form

$$W(z) = -iE_{y\infty}\sqrt{z^2 - b^2}, \tag{1}$$

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